A SEMIOTIC PERSPECTIVE ON LEARNING MATHEMATICS WITH DIGITAL AND ANALOGUE MATERIAL: PRIMARY SCHOOL CHILDREN ACTING ON STATISTICAL DIAGRAMS

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ABSTRACT

This paper focuses on two third-grade students' work on the same statistical question whereby one acts with analogue material and the other with TinkerPlots™. The aim of the research was to find out whether different material influences the actions and, thus, possibly the mathematical interpretations of the learners. To investigate this research interest, a semiotic perspective on mathematical learning according to Peirce was adopted. Based on this perspective, a modification of Mayring's context analysis was made, which allowed the analysis of actions to reconstruct the learners' diagram interpretations. From the analyses, there is evidence that some materials can shorten actions and can automatically establish mathematical relationships and, thus, affect the mathematical interpretations of the learners. At times, however, other actions on different materials can also lead to the reconstruction of the same diagram interpretations. Using these insights, implications for mathematics teaching practice were formulated to assist teachers in selecting materials for designing learning environments to support early statistical thinking.

Keywords: *Statistics education research; Diagrammatic reasoning; Digital and analogue material; Primary school; Early childhood education*

1. INTRODUCTION

To be an empowered citizen in a data-driven community, it is important to work, think and reason with data from an early age (e.g., Ben-Zvi, 2018; Ben-Zvi & Garfield, 2004; Frischemeier, 2020). Leavy et al., (2018) indicated that the "[u]se of appropriate educational tools […], in combination with suitable curricula and other supporting material, can provide an inquiry-based learning environment through which genuine endeavours with data can start at a very young age […]" (p. ix–x). However, the question remains: What material can be used and how should the material be used profitably to create such 'an inquiry-based learning environment' in which young learners encounter endeavours with data? To create such a learning environment as a teacher, it is important to know how learners perform actions on different materials and how each material supports the learners to form mathematical interpretations. To provide such references for practice in mathematics lessons, mathematics education research often analyses or evaluates the material separate from the concrete actions of the learners on the material. In this paper, however, the activity of young learners and, thus, their mathematical interpretations are examined to determine how the learners use the material and, consequently, how and when which material supports the mathematical learning process of the young learners.

This paper reports on whether the mathematical actions of the learners are influenced by different materials (analogue and digital) and whether the potential influence has an impact on the mathematical interpretations of the learners when dealing with a statistical problem. Previous analyses of third-grade students' actions on digital and analogue materials when dealing with a statistical problem (Billion, 2022) indicated that digital materials can act as tools by shortening the learners' actions of sorting data and automatically creating mathematical relationships. This enables the learner to focus on different

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aspects of mathematical processing. Therefore, by analysing the actions of other third-grade students in this paper, the aim is to establish whether these indications can be confirmed.

To investigate this research interest, the research is theoretically framed by a semiotic perspective on statistical learning according to Peirce (1931–1935). This perspective highlights activities on diagrams, which are often creative, inventive, experimental and explorative, as doing mathematics (Dörfler, 2006) and therefore brings the actions of learners to the forefront of interest. This scientific approach fits well with the aim of the paper to investigate the usage of the different materials based on the learners' actions rather than to evaluate the material. Based on the theoretical considerations in the *MatheMat–Mathematical Learning with Materials* study (e.g., Billion, 2022), comparable learning situations with statistical questions were designed, which were used once with analogue material (wooden cubes, data cards, sticky notes, and a square grid) and once with digital material (*TinkerPlots*™, Konold & Miller, 2011). The learning situations can be understood as comparable since the same mathematical relationships can be constructed, even though the materiality differs. To achieve the formulated aim of the paper, the actions of two third-grade students (Nils and Li [pseudonyms]) in the analogue and digital learning situation were analysed when answering the question: *Did more boys or more girls indicate blue as their favourite colour?* For the analysis of the actions, a semiotic adaptation (Billion, 2021a; Billion, 2022) of the context analysis according to Mayring (2014) was used. This enabled the mathematical interpretations of the learners to be described by comparing the actions of the learners and actions resulting from a research-based interpretation of the material arrangement. The reconstructed mathematical interpretations of Nils and Li are compared to find out whether the material influences the actions and possibly also the mathematical interpretations of the learners. This comparison should enable a description of which material supports young learners at a particular step in the mathematical learning process. The findings of this paper in addition to findings from another paper (Billion, 2022) allow the formulation of interpretive hypotheses beyond the individual case for the use of digital and analogue materials in statistics classrooms. These interpretive hypotheses can be used by teachers to make reasoned decisions about when to use which material for designing an inquiry-based environment in which young learners can engage with data.

2. THEORETICAL FRAMEWORK

In this paper, a semiotic perspective according to Peirce (1931–1935) on learning mathematics is adopted. This perspective allows attention to be paid on the actual activities made on the material.

2.1. SEMIOTIC PERSPECTIVE ON LEARNING MATHEMATICS

From the semiotic perspective, learning mathematics is closely linked to working with diagrams. In this way, Peirce (1976, NEM IV) states that "[a]ll necessary reasoning is diagrammatic; and the assurance furnished by all other reasoning must be based upon necessary reasoning. In this sense, all reasoning depends directly or indirectly upon diagrams." (p. 314). Similarly, Dörfler (2006) formulated the importance of diagrams in learning mathematics: "[M]ath is understood (primarily and initially) as a social practice with, on, about, and through diagrams" (p. 105). It becomes clear that in a semiotic sense, mathematical learning is a reflected handicraft with diagrams in a social context rather than something abstract.

The concept of diagrams in the Peircean sense goes beyond the representations of data in a plot and is to be understood in a broader sense than usual. A diagram is a complex sign whose main interest is to represent relationships (Bakker & Hoffmann, 2005). The representation of relationships is constructed by means of a "system of representations" which is defined by a set of rules and conventions (Hoffmann, 2010, p. 42). A diagram can be described as an icon that represents things by imitation and can, thus, express relationships. The complex sign also includes indices whose main function is to direct attention to something and symbols whose meaning is determined by their use, habit or rule (Bakker & Hoffmann, 2005). Most importantly, a complex sign only becomes a diagram if it is used according to the defined rules and conventions (Dörfler, 2006). The usage of signs is a visible activity, for example, actions and gestures. Thus, actions and gestures are essential for mathematical learning and should be the focus of attention.

Reasoning with diagrams in the Peircean sense consists of several steps: constructing a diagram "by means of a consistent system of representations", experimenting with it "according to the rules of the chosen system of representation", reflecting upon the results of experimenting and, finally, to "express these results 'in general terms'" (Hoffmann, 2010, p. 47). Bakker (2004) refers to the importance of describing what is seen in the diagram after manipulating it; this forms the major component of reflecting on statistical diagrams. Looking at the empirical example considered in this paper in which Nils and Li investigate the correlation between the attributes of *favourite colour* and *gender*, diagrammatic reasoning means that the learners construct, in the semiotic sense, a statistical diagram by positioning cases in a plot according to values represented on data cards. By refining or coarsening the two scales of the plot, experiments can be carried out on this diagram and other relationships between the attributes can be recognised through the distribution of the cases. Thus, diagrammatic reasoning consists of a series of processes starting with the observations of the relationships that have come to the fore through experimenting with the diagram, reflecting on them and, as a final step, expressing these observations in general terms. In this, relationships between two attributes can be formulated through the distribution of the cases that are not necessarily expected, and generalisations can be made that are essential for mathematical reasoning.

2.2. ACTING ON DIAGRAMS

Complex signs, as every perceptible sign, evoke one or several ideas in the sign reader (Hoffmann, 2006). Peirce (1932, CP 2.228) termed the idea that is evoked by the sign in the mind an "interpretant", which be expressed as a perceptible sign (a representamen). In this way, "the interpretant can be a reaction to a sign or the effect in acting, feeling, and thinking […]" (Bakker & Hoffmann, 2005, p. 336). Thus, learners express their interpretant in their actions as well as in their gestures (Huth, 2022), which offers the researcher the opportunity to reconstruct the interpretant through the learners' expressions. If the learner has manipulated the diagram, this transformed diagram can be seen as the interpretant of the diagram and, at the same time, as a new diagram (Peirce, 1976, NEM IV). This results in an ongoing sign process that is rarely completed. The interpretant uttered by the learners in their action or construction of a new diagram is determined by the learners' experiences, understanding of concepts, and habits (e.g., Billion, 2021b; Dörfler, 2006; Schreiber, 2013).

As mentioned above, a diagram is a complex sign that is embedded in a habit, and it has to be interpreted according to the conventions that are anchored in the habit. Therefore, learners working with diagrams must first interpret the rules of the diagram to be able to recognise the permissible transformations. The rules and relationships of the diagram define the possible actions on the diagram, but, likewise, they can also restrict those (Bakker & Hoffmann, 2005). If persons are versed in the usage of diagrams, then they recognise that the diagram has certain attributes that are always associated with that diagram. In this context, Peirce (1976, NEM IV) gave the following example: "What is true of the geometrical diagram drawn on paper would be equally true of the same diagram when put on the blackboard?" (p. 317). This means that the material from which the diagram is made does not influence the relationships represented by the diagram. Similar explanations can be found in Dörfler (2015) and Shapiro (1997) who stated the structure of a diagram to be important in that it should describe the relationships between the individual parts rather than focusing on the individual parts which do not affect how they are connected in the structure.

Although the rules and relationships of the diagrams are the same and define the same possible actions, a tool may shorten the actions and distract them from the relationships between the signs by automatically creating multiple relationships triggered by one action (Kadunz, 2016). This means that a tool can automatically establish several mathematical relationships through one action that is performed on the material. Without the tool, the mathematical relationships would have to be established in several individual actions. In this sense, actions are shortened by the tool. Only when the new complex sign generated by the tool is re-interpreted by the learner can the relationship between the perceptible signs and the associated relationship be re-established (Kadunz, 2016). For example, if one draws a line parallel to another line, one must ensure that each point of the new line has the same distance to the existing line (Kadunz, 2016). In the action of drawing, relationships between all the points on both lines are considered. When dynamic geometry software is used, a parallel straight line can be constructed with one click (Kadunz, 2016). In the action of clicking, the relationships between the points on the two lines do not have to be considered. The action of drawing and the relationships to be observed in the action are shortened by the tool (software) to such an extent that no interpretation of the sign is necessary for the action of clicking. Only when the manipulation made by the software is to be interpreted does the connection between the perceptible sign and the relationship between the signs have to be re-established (Kadunz, 2016). A protractor can also be used as a tool to shorten the actions and the relationships that have to be established in the actions. Although a line still has to be drawn, the auxiliary lines on the protractor are parallel to the drawing line, thus, the relationships are created by the tool through the correct application of the protractor. Unlike working with dynamic geometry software, the relationships concerning the line and the protractor must be recognised and established before drawing. It becomes clear in the comparison that the software carries out a greater shortening than the protractor. For this paper, it can be assumed that the software *TinkerPlots*™ also shortens the actions and distracts the actions from the relationships between the signs. In this case, it may not be possible to reconstruct the learners' entire diagram interpretations from the abbreviated actions, but from the activity that the learners carry out after the action to re-interpret the new complex sign. This paper reports on the description and analysis of the actions on different materials. If there is a shortening of the action by a tool, this is not to be judged as better or worse, but it has an effect on when the learners make interpretations so that there is a different focus on the diagram in the sense of diagrammatic reasoning according to Peirce (1931–1935).

3. RESEARCH FOCUS

From the theory section, it becomes clear that to be able to act on a diagram, the individual must recognise the relationships beyond the diagram and observes those relationships when making manipulations on it. In the semiotic sense, the manipulations made on the diagram can be seen as the learners' interpretants. For this paper, this means that by analysing the actions performed on the material arrangement, it is possible to reconstruct the learners' interpretations of the arrangement as a mathematical diagram. However, the analysis of gestures also contributes to the reconstruction of an approximate complete diagram interpretation of the learners. As Huth (2022) and Chen and Herbst (2012) have found, gestures can indicate relationships between signs, generate mathematical ideas and, thus, can become diagrams themselves. A gesture can be described as an explicit communicative act addressed to someone else and guided by a specific intention (Kendon, 1984). Actions, on the other hand, are part of the performance of a specific task, such as the manipulation of things in the environment as part of the purposeful activity (Kendon, 1984). Nevertheless, it is sometimes difficult to decide whether an action does not also have a communicative purpose. Therefore, this paper does not attempt to separate gestures and actions from each other but to include their interplay in the analysis.

Looking at the diagrams that Nils and Li are working on to answer the given questions, which are realised with analogue and digital material, it can be assumed that the material plays a subordinate role as the relationships between the parts of the diagram are of greater importance than the materiality of the signs. It can be assumed that the learners have to observe the same relationships between the signs in their actions to be able to answer the question about the correlation between *favourite colour* and *gender* using a bivariate plot that they have constructed. However, if the material functions as a tool and shortens the actions, it may be that the relationships between the signs no longer need to be established in the actions as the tool establishes these relationships automatically, triggered by a short movement. This could affect the way that the learners interpret the diagram and, thus, the reconstruction of the learners' interpretations of the diagram.

3.1. RESEARCH QUESTIONS

Based on these theoretical considerations and extending the findings described from another paper (Billion, 2022), the following research questions arise for the comparability of the findings:

(1) Which mathematical interpretations of the learners can be reconstructed from the actions on the complex signs realised with digital and analogue material?

(2) What possible differences can be described between the learners' reconstructed interpretations due to the materiality of the signs, when the mathematical relationships between the signs are the same?

4. METHODOLOGY AND STUDY DESIGN

In the following, the design, data collection and data preparation of the *MatheMat* study are discussed. In addition, the semiotic adaption of the context analysis according to Mayring (2014) is described in detail.

4.1. STUDY DESIGN

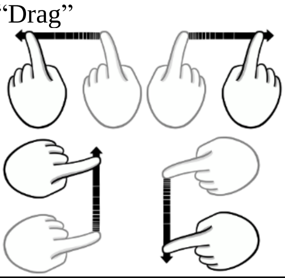
The study, *MatheMat–Mathematical Learning with Materials,* focused on the actions of primary school children while working on geometrical and statistical tasks with digital and analogue material. The study aimed to reconstruct the mathematical interpretations of learners based on their actions on the material to investigate the potential influence of the different materials on the learners' mathematical interpretations (Billion & Vogel, 2021). For this purpose, the learning situations were designed with different materials (digital and analogue), whereby the same mathematical relationships between the parts of the material arrangement can be established. Overall, 32 learners worked in pairs on the learning situations designed. The pairs of learners each worked on one statistical and one geometrical task, once with digital and once with analogue material. The learners' processing was videotaped so that every action, gesture, or phonetic utterance was available for analysis.

4.2. DATA COLLECTION AND PREPARATION

Four pairs of learners work on the learning situation considered in this paper in which the learners were to answer given questions with bivariate plots constructed by themselves. To create a bivariate plot, two pairs used *TinkerPlots*™, while the other two pairs worked with wooden cubes, data cards, sticky notes and a square grid. The learners' processing of the statistical task was recorded with two cameras. One camera focused on the actions and gestures on the material, while the other camera recorded the overall scene. From the recordings, comparable passages were singled out for transcription. These passages are characterised by the fact that the learners have worked with different materials on the same statistical question at the same point in the process. This means that only learners who worked on the same statistical question at the same time but with different material were selected for the analysis. In this case, it was Nils and Li both with their respective work partners. The learners Nils and Li were specifically selected for the analysis because their actions have already been analysed in a geometric learning situation (Billion, 2021a). In this way, comparability of the analysis findings is possible across topics. In a semiotic sense, it can be assumed that the learners show a comparable use of the potential diagram in the selected passages. Here, it is possible to compare whether the actions with the analogue and digital signs differ, although the same relationships between the signs may need to be observed in the actions. It can also be investigated whether different actions allow conclusions to be drawn about different diagram interpretations. The transcribed video passages formed the starting point for the qualitative analysis. Initially, the focus was only on the transcribed passages, then, in the course of the analysis, the entire processing of the learners' actions captured on video was included.

To analyse the actions on the digital and analogue material as accurately as possible, the learners' movements were described as precisely as possible during transcription. It is important to note that even with actions on the analogue materials, it is difficult to distinguish between actions and gestures (e.g., Andrén 2010; Harrison, 2018; Kendon, 1984; Vogel & Huth, 2020). In the case of actions on digital material, this distinction becomes even more difficult and is sometimes not possible. In this paper, therefore, all movements across the surface of the computer screen and the resulting manipulations of the software used were noted as actions. To describe these actions, parts of the *Touch Gesture Reference Guide* (Villamor et al., 2010) were used and adapted for *TinkerPlots*™. Table 1 (Billion, 2022) shows the movements on the screen, their descriptions and the resulting manipulations and relationships, which were established by *TinkerPlots*™. The movements were illustrated by Petra Tanopoulou (reproduced and adapted from Villamor et al. [2010]).

Table 1. Description of the movements and the resulting manipulations

Movement on the screen	Description of the movement	Manipulations and relationships
"Tap" 	"Briefly touch surface with fingertip" (Villamor et al., 2010)	Select an attribute and transfer the values to the dots in the plot. *TinkerPlots*™ establishes a relationship between each data card and the matching dot in the plot.
"Drag" 	"Move fingertip over surface without losing contact" (Villamor et al., 2010)	Sorting the dots in the plot according to a specific attribute. *TinkerPlots*™ establishes a relationship between an attribute and an axis, between the values plotted on an axis, and between the values and the dots in the plot.

4.3. ANALYTIC METHODS

For the reconstruction of the learners' mathematical diagram interpretations, a semiotic specification (Billion, 2023) of the context analysis (Mayring, 2014; Vogel, 2017) was applied. With the help of this analysis, the learner's interpretant was compared with an interpretant formulated based on current research to be able to describe the learner's diagrammatic interpretation.

As described in the theory section, each sign evokes an interpretant and this interpretant can be seen as a reaction to the sign, such as an action, a feeling, a thought, or a gesture. Unlike the spontaneous reaction of learners to a sign, the "'final logical interpretant' […] comes out ideally 'in the long run' of scientific communication" (Bakker & Hoffmann, 2005, p. 336). This final logical interpretant cannot be formulated because it is idealistic, therefore, an interpretant based on current research is formulated that comes close to this ideal (Billion, 2023). The interpretant based on research includes the description of relationships between the signs and the actions on the complex sign that are possible from these relationships (Billion, 2023). Only relationships and the resulting actions that are important for solving the task are described, although it is important to note that the learners may have recognised other relationships between the signs and performed actions based on those relationships.

The analysis consists of several explications. In each explication, by comparing the learner's actions with the actions of the research-based interpretant, the interpretations made by the learner for his or her actions are reconstructed (Billion, 2023). By adding more and more of the learner's actions and contrasting them with the research-based interpretant across the explications, the learner's diagrammatic interpretations were reconstructed throughout the processing. In Explication 1, a small transcript passage was selected in which the learner acted on the complex sign. This small transcript passage was compared with the research-based interpretant to describe the learner's diagram interpretation. In the further course of the analysis, similar or identical actions of the learner found in the transcript were selected in order to relate them again to the research-based interpretant (Explication 2). In this way, the learner's interpretations reconstructed in Explication 1 can be extended, confirmed or rejected. In Explication 3, non-transcribed passages of the learner's identical or similar actions in the learning situation are selected to compare them with the research-based interpretant. In the comparison, the learner's further interpretations of the diagram are described.

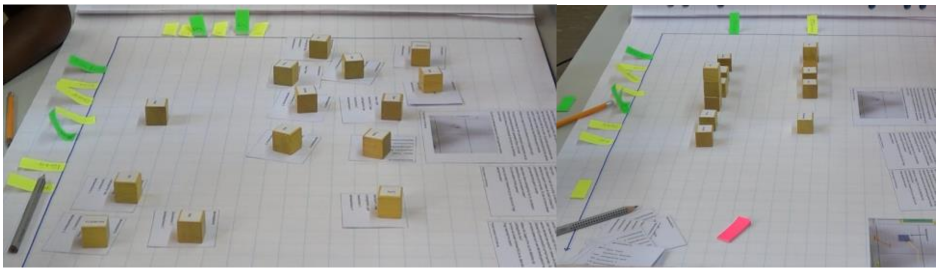
5. EMPIRICAL SNAPSHOT

In the following, the learning situation is described in which the learners investigated the relationship between the attribute *gender* and *favourite colour*. More specifically, the following question was answered by the learners: *Did more boys or more girls indicate blue as their favourite colour?* Friel et al. (2001) distinguished three different levels at which the understanding of a plot can emerge: at the first level, learners extract data from the plot, at the second level learners interpret relationships between the data as shown in the plot and, at the most advanced level, learners explore relationships that are implicitly evident in the plot. The question that the learners dealt with for this paper can be assigned to the two more advanced levels, in contrast to other research in the *MatheMat* study (Billion, 2022). In this case, the learners were required to identify relationships between the values of two attributes, compare the number of cases, and reduce the relationships identified in the data to a statement.

5.1. THE LEARNING SITUATION

The learning situation dealt with bivariate plots of nominal and ordinal data. The learners were to investigate data from 14 second-grade students. At the beginning of the learning situation, the learners explored the analogue and digital materials freely. In this way, they had the opportunity to interpret relationships between the parts of the material arrangement themselves. Afterwards, the researcher explained what could be investigated mathematically with the material and introduced the statistical questions the learners were to answer. The children had different statistical questions in front of them and were free to decide which question they wanted to answer. In this paper, the focus is on the sections of the processing in which the learners dealt with the question of whether more boys or more girls indicated *blue* as their *favourite colour*. The data were provided on data cards that included five attributes: name, gender, favourite colour, grade in mathematics, and grade in German (see the digital version in Figure 2a). The learners worked with data cards as it has been determined that learners produce more complex and informative representations by working with these cards than when using drawing representations (Harradine & Konold, 2006). In particular, the representation of multivariate data is easier to implement (Harradine & Konold, 2006). Based on the digital or analogue data cards, the learners can organise analogue cubes or dots in *TinkerPlots*™ "in small steps using simple actions such as separating them into groups and ordering them according to the value of one of the attributes" (Harradine & Konold, 2006, p. 4). Due to these simple actions, learners can easily reorganise the representations to create more informative representations.

Analogue materials. Nils and his partner worked with analogue material. The two learners had analogue data cards, cubes marked with the names of children (see Figure 1a), a square grid with squares having the same size as the sides of the cubes, and sticky notes to label them with different values (see Fig. 1a-b). To sort the cubes according to the attributes of *favourite colour* and *gender*, sticky notes must first be labeled with the relevant values of the two attributes. The learners then had to decide which attribute they wanted to mark on which axis. In Figure 1b, the attribute *favourite colour* was plotted on the x-axis and the attribute *gender* on the y-axis. To sort a cube by two attributes, the relevant values of the attributes had to be read from the data card and the cube positioned according to the values in the plot. When all the cubes were positioned in the plot, it showed that three boys and no girls chose *blue* as their *favourite colour*.

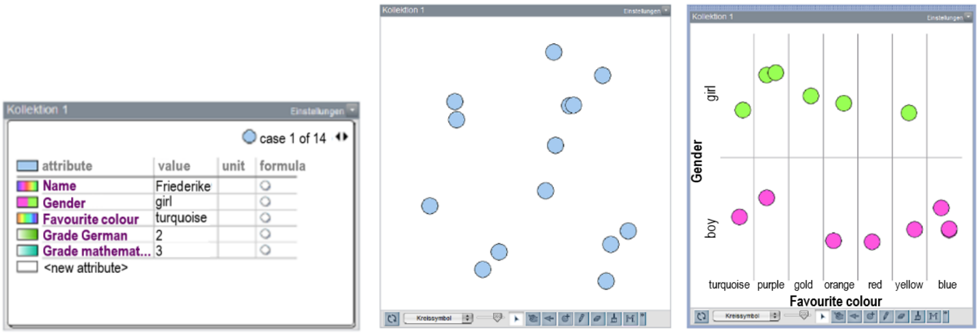


(a) (b)

Figure 1a-b. Analogue material used to sort the data according to two attributes

Digital materials. To solve the given question, *Did more boys or more girls indicate blue as their favourite colour?* Li and her partner were provided with *TinkerPlots*™. *TinkerPlots*™ is a software tool for simulating and visualising data (Frischemeier, 2018; Harradine & Konold, 2006). The software was set up to show the data cards in which the values of the 14 students were entered (see Figure 2a). In addition, a plot was opened in which the data were represented as dots. The dots in the plot were coloured blue at the beginning of editing (see Figure 2b). When an attribute is tapped on the data card, the dots in the plot change colour according to the value of the attribute (see Figure 2c). For the attribute *favourite colour*, *TinkerPlots*™ did not colour the dots in the plot according to the given favourite colour but coloured them independently of the values on the data card (see Figure 5). Thus, the learners could not rely on the colour of the dots but had to relate the value on the scale to the distribution of the dots. The researcher drew attention to this in the introduction to the material.

To sort the dots in the plot by two attributes in *TinkerPlots*™, an attribute must first be selected from the data card and then moved horizontally or vertically using a drag movement. If a horizontal drag movement is made across the plot, the values of the selected attribute are plotted on the x-axis. If a vertical drag movement is made, the attribute is plotted on the y-axis. The software simultaneously positions the dots in the plot according to the scaling. If an attribute is plotted already on the x-axis, the second attribute can be selected from the data card. Once selected, the dots can be dragged vertically to plot the attribute on the y-axis. In Figure 2c, the attribute *favourite colour* has been plotted on the x-axis and the attribute *gender* on the y-axis; *TinkerPlots*™ has positioned the dots according to the scale automatically. In Figure 2c the dots are coloured according to the attribute *gender*.



(a) (b) (c)

Figure 2a-c. Digital material with which sorting can be carried out according to two attributes

5.2. SEMIOTIC PERSPECTIVE ON THE LEARNING SITUATION

In the semiotic sense, the data cards and the bivariate plots in *TinkerPlots*™ can be interpreted as a diagram. Therefore, the analogue material was chosen based on the features of *TinkerPlots*™ in such a way that the data cards and the wooden cubes, with which a bivariate plot can be produced, can be interpreted as a diagram. This allows learners to work with comparable diagrams in both the analogue and the digital material. To describe these diagrams appropriately, the idea of a *model of* and *model for* by Gravemeijer (1999; 2002) is used. This approach focuses on emergent models that are created through an ongoing sign process. In this paper, the transition from *models of* context-specific situations to *models for* formal argumentation in the ongoing sign process is transferred to the work with diagrams in the semiotic sense according to Peirce (1931–1935). The diagram *data card* can be seen as a *model of* different measurements and surveys. To create a bivariate plot from the data cards, an ongoing sign process, that is, different manipulations on the material, is necessary. By positioning the cases in the plot, the relationships between the attributes become the focus. The position of the cases expresses how the values are related to a local position that depends on the scaling used. The positioning of all the cases displays the correlation or association of the data, which can be described across the individual values. If the positioning of the cases is recognised as a distribution, the context moves into the background and the relationships between the two attributes become the focus.

The actions on a plot can be compared to actions made on data cards. Positioning the cases in the plot is equivalent to sorting the data cards according to the values of the two attributes. Since the actions described are equivalent, the diagram *plot* can be interpreted as a *model of* the diagram *data card*. When the learner uses the diagram *plot* to talk about the relationship between two attributes that is made clear by the distribution, the diagram *plot* becomes a *model for* mathematical reasoning. The plot helps the learner deal with the relationships between attributes and derive a general meaning of the relationships through the positioning of the dots/cubes in the plot. Learners can start to see the distribution as a whole and make connections. Through this change from a *model of* to a *model for*, creating a plot and acting with it can become the intermediate between the concrete values of attributes and the general relationship between attributes. In this way, the number as a measurement or survey on the data card becomes a position in the plot and can, thus, be understood as being integrated into a network of relationships.

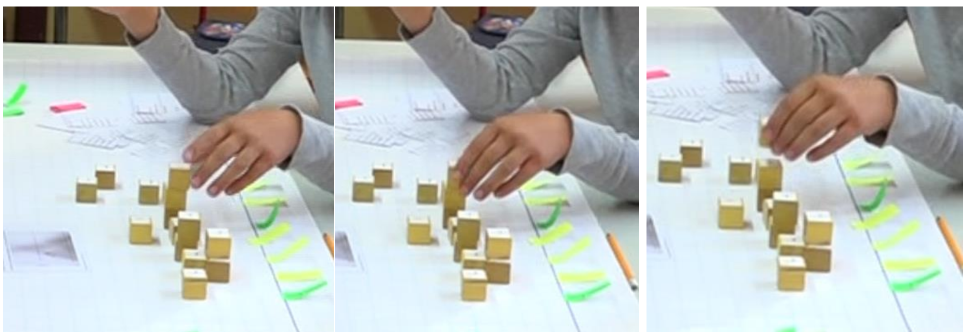
The transition from informal to formal mathematics, or the transition from a *model of* to a *model for*, is particularly important for researchers and teachers. For learners, this distinction is not necessarily of interest when working with diagrams. Following this line of reasoning, teachers or researchers can recognise with their trained eye whether the diagram is being used by the learner as a *model for* thinking about connections or remains a *model of* the data card. In this paper, the qualitative analysis of the learners' actions refers to how the learners use the diagrams.

After a closer look at the diagram *data card* and the diagram *plot*, it can be assumed that when working with digital or analogue material the same relationships have to be recognised to create a bivariate plot based on the values of the data card. For example, relationships between the axes and the values of an attribute must be established, or the relationship between an individual case and the values on the axes. In both the digital and the analogue material, the focus of the data card is initially on the individual; the data card is rather a *model of* different measurements than a *model for* mathematical reasoning. By creating the plot, the view can then be directed more towards the distribution of all cases instead of one individual. Therefore, the plot can more easily become a *model for* mathematical reasoning. Following Dörfler (2015) and Shapiro (1997), it can be assumed that the same relationships have more influence on the interpretation of the learners than the different materiality of the signs.

5.3. ANALYSIS OF ACTIONS ON ANALOGUE MATERIAL

The reconstruction of Nils's diagram interpretations began with the following transcribed actions. Nils and his partner were asked to answer the question of whether more girls or more boys selected *blue* as their *favourite colour*. Nils worked on similar questions previous to the transcribed passage. During the qualitative analysis, identical and similar actions, gestures, and phonetic utterances of Nils were integrated for the reconstruction of his diagram interpretations. The steps of the analysis (Explications) were summarised due to page length restrictions.

Nils: *Moves the left hand to the cubes with the value two of the attribute grade in mathematics and the value blue of the attribute favourite colour* (see Figure 3a).
These cubes are labeled Ray, Can, and Ogan.
Grabs with his left hand the top cube (see Figure 3b).
Lifts the top cube with the left hand about 3 cm high (see Figure 3c).



(a) (b) (c)

Figure 3a–c. Nils's actions on the analogue plot showing a sorting according to the attributes "grade in mathematics" and "favourite colour"

Explication 1. To answer the question asked, it was necessary to sort the cubes according to the attributes of *favourite colour* and *gender*. For such a sorting, many relationships must be recognised. For the research-based interpretant, the relationships between the signs and the resulting actions were formulated (Points 1–4). During the analysis, the research-based interpretant was contrasted with Nils's actions. The research-based interpretant is summarised and the four main relationships and the resulting actions are stated:

1. *Relationship between the data card and the cube:* the learners have to recognise that one cube matches one data card. In this way, learners can later transfer the values on the data cards to the cubes to sort them. To establish this relationship, the learners can match the cubes to the corresponding data cards (see Fig. 1b).
2. *Relationship between the attribute and the axis:* the learners have to recognise that one attribute can be plotted on one axis. Depending on which axis the first attribute is plotted, the second attribute must be plotted on the other axis. To establish such a relationship, learners need to label the axes on the square grid with their respective attributes.
3. *Relationship between the values on the axes:* learners need to recognise that all values of an attribute can be plotted on the axis marked with the corresponding attribute. To establish this relationship, learners have to transfer the values from the data cards to the sticky notes. The labeled sticky notes must then place next to each other on the axis. Since both attributes are nominal, the values do not have to position on the axis in a specific order. Nevertheless, leaving the same distances between the values increases the readability of the plot.
4. *Relationship between the values on the two axes and the cube:* to sort the cubes, learners establish relationships among the values on both axes and the cube. To establish such relationships, it would make sense to move a finger vertically and horizontally, starting from the desired value on each axis, whereby the meeting of the fingers marks the position of the cube. Thus, to establish the relationships, the values of the cubes must first be taken from the data cards.

In comparison to the interpretant based on research, it becomes clear that Nils did not sort the cubes according to the attribute *favourite colour* and *gender*. He selected the stack of cubes consisting of all the children who have indicated *blue* as their *favourite colour* from the sorting by *grade in mathematics* and *favourite colour* in front of him. As Nils performed a purposeful action towards the cases who indicated *blue* as their favourite colour after reading the statistical question, it can be assumed that Nils could interpret the relationship between the value *blue* of the attribute *favourite colour* and the cubes in the plot in front of him. It can also be assumed that Nils wanted to look at the names on the cubes to check whether they are girls' or boys' names. Following this assumption, he included the second attribute of *gender,* which was relevant to the question to be answered. Thus, he decided not to create a plot with the attributes that fit the answer to the question but used the already existing plot. Nils did not create another plot. This plot, if created, would still be a *model of* the data, which could have only become a *model for* mathematical reasoning after completion. Likely it was easier for Nils to use the plot that already existed as a *model for* answering the question and, thus, for mathematical argumentation. The diagram *plot* created already helped him to find relevant data and to think about new relationships among the data and other attributes.

Explication 2. Nils repeated the actions taken in Explication 1 to lift the second cube from the stack. With this and other movements, the reconstructed diagram interpretation can be confirmed.

Explication 3. As there are many passages where Nils acted on the diagrams, Nils's reconstructed diagram interpretation can be expanded. By matching the cubes to the data cards, it was assumed that he was implying that both the name on the cube and the name on the data card represent the same child. This supports the assumption that Nils recognised a relationship between the cubes and the data cards. In a further passage, Nils assigned a hypothetical attribute to the x-axis and the y-axis by suggesting to write the value *boy* on the x-axis and *colour* on the y-axis. It can be assumed that Nils established a relationship between the attributes and the axes. In further passages of the processing, Nils assigned all values of the attribute *favourite colour* to the x-axis by sticking all sticky notes on which different colours were written to the x-axis at approximately the same distance apart. He assigned all values of the attribute *gender* to the y-axis. Since the values of the two attributes were nominally, there was no evidence Nils established a relationship between the values on the axes. By including further passages where he assigned values of an ordinally scaled attribute to an axis in an ordered manner, it can be assumed that he can establish this relationship in his actions. In addition, passages were found where Nils simulated the positioning of a cube. To do this, he moved the index finger of his right hand to the right on the square grid, at the same height parallel to the x-axis, and moved his left index finger upon the square grid, at the same height parallel to the y-axis, so that his index fingers meet. Through these actions, it can be reconstructed that Nils established a relationship between the values of the x-axis and the y-axis. By tapping at the meeting point of his fingers on the square grid, it can be assumed that Nils marked the meeting of his fingers in the position of the cube. In other passages, Nils sorted the cubes according to the attributes of *gender* and *favourite colour*. He translated, in each case, two values on the diagram *data card* into the positioning of the cube in the diagram *plot*. Various passages suggest that Nils looked at the data cards before he positioned the cubes in the plot. By looking at the data card, it can be assumed that he recognised the values on the data card and expressed them by placing the corresponding cube in the diagram *plot*. The translation turns the diagram *plot* into a *model of* the diagram *data card*. Nils used the plot as a *model for* mathematical considerations and statistical reasoning.

5.4. ANALYSIS OF ACTIONS ON DIGITAL MATERIAL

Li and her partner worked on the same question as Nils and his partner. The reconstruction of Li's diagram interpretation begins with the following transcribed gestures and her phonetic utterances. In the further course of the analysis, Li's phonetic utterances, her gestures and her actions in which she deals with the relationship between the attributes of *favourite colour* and *gender* were included. Due to the lack of space, a summary of the reconstructed diagram interpretation was made in each step of the explications.

Li: *Removes the right forearm from the tabletop.*
Rests her right elbow on the table (see Figure 4a).
Moves the right hand towards the tablet with the index finger extended (see Figure 4b).
Holds the right index finger about 5 cm above the tablet (see Figure 4b).
The index finger is above the plot and points in the direction of the value "blue" and "boy" (see Figures 4b and 5).
"Here are three guys"
Raises the right hand about 2 cm upwards (see Figure 4c).

(a) (b) (c)

Figure 4a–c. Li's gestures above the plot

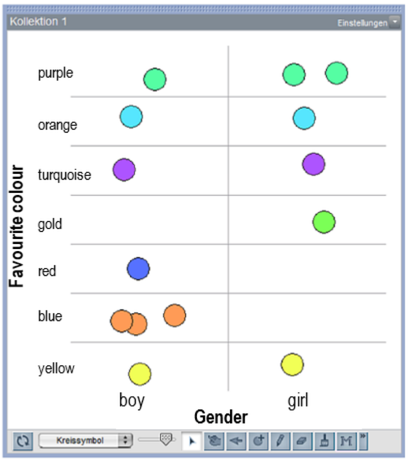


Figure 5. The positioning of the dots in the plot while Li gestures

Explication 1. To answer the question posed, it is useful to sort the dots in the plot according to the attributes of the *favourite colour* and *gender*. The relationships between the signs to be recognised and the resulting actions on the diagrams are described in the research-based interpretant. As in the case of working with analogue material, there are four main relationships in the foreground:

1. *Relationship between the data card and the dots in the plot:* in order to sort the data, the learners need to realise that the dots in the plot must take the value of the attribute to be sorted by. To establish this relationship, it is necessary to select an attribute with a tap movement on the data card. Triggered by the tap movement, all the dots in the plot automatically take on the value of the selected attribute, whereby in the case of the favourite colour, other colours are chosen by the software than the values on the data cards.
2. *Relationship between the attribute and the axes:* the learners recognise that one attribute has to be plotted on the x-axis and the other on the y-axis. To establish a relationship between an attribute and the x-axis the learners perform a drag movement horizontally across the plot. A relationship between an attribute and the y-axis is established with a drag movement vertically across the plot.
3. *Relationship between the values on the axes:* to establish a relationship between the values on one axis the learners need to make a drag movement starting from a dot horizontally across the plot. The software automatically performs a suitable scaling on the x-axis. A drag movement vertically across the plot triggers a scaling on the y-axis.
4. *Relationship between the values on the two axes and the position of the dot in the plot:* to be able to sort by two attributes, a relationship must be established between the values on the two axes and the dot. The learners perform a drag movement starting from one dot (vertically or horizontally) across the plot. The software automatically positions the dots according to the scaling of the axis. Once all the dots are positioned according to the scale on the x-axis (Attribute 1), the learners perform a drag movement vertically and the software positions the dots according to the scale on the y-axis (Attribute 2).

By comparing Li's interpretant with the research-based interpretant, it can be seen that Li did not act on the diagram *plot* but referred to it verbally and through gestures. The position of her index finger and the utterance "Here are three guys" can refer to her recognition that the boys who indicated *blue* as their *favourite colour* were found at that point in the plot. Regarding her phonetic utterances and gestures, Li most likely recognised that the dots in the plot were already positioned appropriately for the given question, and that she was able to read off the children who have indicated *blue* as their *favourite colour*. She, thus, established a relationship between the values on the two axes and the position of the dots in the plot. Li did not start from the colour of the dots in the plot (the dots representing boys who chose *blue* as their *favourite colour* are shown in orange) but from the scaling of the y-axis. She also recognised that three dots are assigned to the value *boy* on the x-axis. It can be assumed that Li was able to interpret the diagram *plot* because she recognised a relationship between the position of the dot and the values on the axes. The diagram *plot* as a *model of* the data card, thus, becomes a *model for* statistical reasoning.

Explication 2. The reconstructed diagram interpretation of Explication 1 can be confirmed since Li gestured above the plot where it showed the girls indicated *blue* as their *favourite colour*. Based on Li's phonetic utterance "… and there are no girls here", it can be assumed that she recognised that no girl indicated *blue* as her *favourite colour*. She again established a relationship between the two axes and the distribution of the dots in the plot. The following gestures can be used to reconstruct that Li wanted to compare the boys and the girls. The spoken conclusion that more boys like blue confirmed the reconstructed diagram interpretation. Explication 2 shows that Li used the diagram *plot* as a *model for* reasoning about the distribution of the dots. In Explications 1 and 2, Li's diagram interpretation was reconstructed exclusively from her gestures and phonetic utterances. Actions on the diagrams were not recognisable.

Explication 3. At the beginning of the processing, Li made rapid successive tap movements with which she selected several attributes on the data card and, thus, the software simultaneously colours the dots in the plot. Since she did not subsequently sort the dots in the plot, it can be assumed that Li could not interpret the relationship between the diagram *data card* and the diagram *plot*. Based on Li's later tap movements, it can be reconstructed that Li specifically selected an attribute and used it for sorting according to the values of the attribute. In these passages, it can be reconstructed that she was able to interpret the relationship between the data card and the plot, which can now be regarded as a *model of* the data card. In addition, other passages were found where Li performed a drag movement across the plot, starting at a purple dot to the left of the plot. Based on the direction in which the drag movement was executed, it can be reconstructed that Li established a relationship between the attribute and the x-axis. Based on Li's actions on the plot, *TinkerPlots*™ made a scale on the x-axis and positioned the dots according to this scale. Thus, it cannot be reconstructed from Li's actions whether she was able to interpret the relationship between the positioning of the dots and the values or the relationship between the values on the scale. Only through the following actions, in which Li performed another drag movement in the same direction several times in a row, it can be assumed that she interpreted the scaling and realised that it still needed to be refined. Further gestures and phonetic utterances illustrated that Li recognised the relationship between the position of the dots and the values on the scale. It was reconstructed through her gestures and spoken language that Li recognised the different relationships in the plot. A reconstruction based on her actions was not possible because one action initiated several manipulations in the software. This means that *TinkerPlots*™ established relationships automatically without separate actions from the learner. In this way, *TinkerPlots*™ functions as a tool and shortens the actions by establishing relationships automatically. By shortening the actions and establishing relationships, the plot changed quickly from a *model of* the data card to a *model for* mathematical reasoning. Li, however, must be practised in interpreting the relationships automatically created by *TinkerPlots*™. After all, other passages showed where Li's gestures and spoken language allowed for a different reconstruction. Those passages showed that Li only recognised the relationship between the positioning of the dots and one axis and neglected the relationship with the other axis. Thus, she did not consistently succeed in interpreting the positioning of the dots that *TinkerPlots*™ made.

5.5. COMPARISON OF THE RESULTS FROM THE ANALYSES

Overall, the reconstructed diagram interpretations of Li and Nils are similar. It was determined from the reconstruction of their actions that both students were able to establish a relationship between the diagram *data card* and the diagram *plot* and a relationship between the attributes and the axes. With the inclusion of the learners' gestures and phonetic utterances, it was reconstructed for both that they were able to establish a relationship between the values on one axis and between the values on both axes and the position of the cases. Furthermore, they use the diagram *plot* initially as a *model of* the data card and later as a *model for* statistical reasoning to answer the given question. It was shown that despite different materials, similar diagram interpretations were reconstructed, whereby it became obvious that by shortening the actions through the digital material the diagram *plot* can be used quickly as a *model for* mathematical reasoning by the learners. This analysis result was also reflected in the processing time, as Nils and his partner who worked with the analogue material needed about 41 minutes to process the learning situation, and Li and her partner who worked with the digital material worked for about 30 minutes on the learning situation. When using analogue material, the learners have to perform the manipulations themselves and, during the manipulation, the diagram *plot* is still a *model of* the data card. On this basis, it was shown in Nils's analysis that he used a plot that did not fit the question to answer the question. This may indicate that he used the diagram *plot* as a *model for* mathematical reasoning and skipped the process of manipulation where the plot was still a *model of* the data card. He succeeded in this because he understood a cube represented a boy or a girl by looking at the names on the cubes.

A closer comparison shows that Nils's interpretation of the diagrams can only be reconstructed based on his actions. This is not always the case with Li. It became clear that Nils established the relationships of the statistical diagrams in separate actions. Since Li worked with *TinkerPlots*™, the software abbreviated her actions as a tool and automatically established some relationships. In this way, the relationship between the values on an axis and the relationship between the values and the dots were established automatically by one drag movement. Conversely, it is not possible to reconstruct from Li's actions whether she was able to interpret the relationships because she did not establish them in separate actions. It was, however, recognisable from Li's gestures and phonetic utterances that she interpreted the relationships made by the software (see Explication 1). This means that she did not have to establish the relationships in her actions: she re-established them by interpreting the signs made by *TinkerPlots*™. In this way, there was a shift from expressing the relationships in the actions to expressing them in gestures and phonetic utterances. This shift means that with the analogue material, the interpretation of the diagram took place during the manipulation of the diagram, whereas with the digital material, firstly, the software generated the sign and then the result of this sign generation was interpreted by reconstructing the diagrammatic relationships. However, due to the abbreviation of the actions, the re-interpretation of the mathematical relationship was not always easy, thus, in Li's analysis, there are places where she did not recognise the relationships between the signs.

6. CONCLUSION AND DISCUSSION

This paper aimed to reconstruct the learners' diagram interpretations through the actions on diagrams that represented the same mathematical relationships but were realised with different materials. Furthermore, the question of whether the material influenced the reconstructed diagram interpretations was considered, even though the same mathematical relationships were shown in the diagrams. In the following, the main results of this paper are discussed and based on these, implications for teaching practice in mathematics lessons are formulated to aid teachers when choosing materials for supporting early statistical thinking.

6.1. MAJOR FINDINGS

The analyses showed that at some points in the mathematical learning process, despite different actions on the different materials, the same diagram interpretations could be reconstructed. In terms of semiotic theory, this means that the appearance of the signs and the appearance of the actions do not affect the learners' interpretations. Thus, appearance is subordinate to the mathematical relationships that are established by the actions of the learners. The mathematical interpretation of the learner is dependent on the mathematical relationships that are established with the material and less on the look or feel of the material. These findings are in line with previous analyses in which the learners Nils and Li worked on a geometric learning situation (Billion, 2021a). In contrast to the examples in this paper, Nils worked with digital material and Li with analogue material. The interpretive hypothesis that can be derived from the findings from the analyses is that the learners' interpretation depends on the relationships they establish and not on the look or feel of the material, which may also apply to engagement with other mathematical topics.

Furthermore, in the comparison of the reconstructed diagram interpretations, a shift between the expression of relationships by actions (analogue material) to an expression of relationships by language and gestures (digital material) becomes apparent. In contrast to analogue material, working with digital material leads to a shortening of the actions as the software automatically establishes the relationships as a tool. In this way, the relationships between the parts of the digital material arrangement do not have to be established in the actions, but the relationships created by the software have to be interpreted afterwards. These findings are consistent with results from analyses of actions on different materials in another statistical learning situation (Billion, 2022).

By comparing the results with the Peircean semiotic view of mathematics learning, it can be shown that through the material different steps of diagrammatic reasoning become focused. When working with analogue material, the relationships between the parts of the material arrangement must be considered, recognised and interpreted during the construction and manipulation of the diagram. Concerning the steps of diagrammatic reasoning, the focus is on creating and manipulating diagrams. When working with digital material, however, the relationships between the parts of the material arrangement do not have to be interpreted during the actions. Instead, these relationships must be re-established if the material arrangement produced by the software is to be interpreted. In this way, when working with digital material that shortens the action, the focus is on the other steps of diagrammatic reasoning: reflecting on the results of manipulating the diagram and formulating these results in general terms. This different focus on the diagram also affects how the diagram *plot* is used by the learners. In analogue material, the learners themselves construct and manipulate the diagram in their actions, thus, it can be seen as a *model of* the data card for a longer time in the action process. Only after manipulation can the diagram be used as a *model for* mathematical reasoning and for answering the question. By focusing on the reflection of the manipulations through the digital material, the diagram *plot* can become a *model for* mathematical reasoning after only a short move across the screen. If this result is related to the question that the learners were asked to answer in the analysed example, it appears that questions that can be assigned to a higher level according to Friel et al. (2001) could be answered seamlessly, as the shortening of actions helps to focus more quickly on relationships among data and the distribution of the data. In this way, the diagram *plot* can be used quickly as a *model for* statistical reasoning. Compared to other research in which the learners were asked to answer questions that could be assigned to a lower level (Billion, 2022), it is conjectured that the difficulty of the question also has an impact on the choice of material. This result is supported by the fact that Nils used a plot that he had already constructed to answer the question, even though it did not optimally fit the question he was supposed to answer. Probably, in this case, the fact that he could use this plot as a *model for* statistical reasoning outweighs the fact that the plot was not sorted according to the appropriate attributes. To be able to interpret a sign produced by a tool and to recognise the relationships among the parts of the new material arrangement, learners must become practised with the use of the diagrams.

6.2. IMPLICATIONS FOR USING MATERIALS TO SUPPORT EARLY STATISTICAL AND PROBABILISTIC THINKING

The findings of this paper can be used to help teachers make informed decisions about when to use what material to create an 'inquiry-based environment' in which young learners can engage with data. Since in the mathematical learning process the appearance of the actions and the materiality of the signs are subordinate to the mathematical relationships established by the learners, it is important for teachers not to make decisions for or against a material based on the appearance or feel of the material. Rather, choices should be based on three items identified as important for the choice of material to design a learning environment (Billion, 2021a; 2022).

Firstly, the choice of material should largely depend on the learners' experience with statistical material arrangements. If young learners are not yet familiar with working with data, it makes sense to use analogue materials at the beginning, where from the start they have to consider relationships in separate actions during the process of sorting the data. Of course, digital materials can also be used which do not function as a tool, so that there is no shortening of the actions and the relationships between parts of the material must be established when acting on these materials. In this way, learners need to establish the relationships between the parts of the diagram themselves and are then likely to be able to recognise and interpret them later when a material abbreviates the actions and establishes the relationships automatically.

Secondly, the learning objective plays a central significance in the selection of the material. If the focus is on constructing and manipulating diagrams when dealing with data, then care should be taken to ensure that the learners attain the mathematical relationships independently by their actions. If the focus is more on reflecting on the results of the manipulations on the diagram, shortening the learners' actions can support the learners to focus on interpreting the results. Such a focus can be used to identify relationships among large amounts of data and to address the distribution of the data.

Thirdly, it becomes clear in comparing the findings across papers (Billion, 2021a; 2022) that the statistical question to be answered by the learners also plays a role in the selection of the material. If the question can be assigned to a more complex level, it is advisable to use a material that shortens the actions so that the learners can use the diagram quickly as a *model for* statistical reasoning. As mentioned earlier, the diagram *plot* is used as a *model of* the data card during its manipulation and construction. Subsequently, once the results of the manipulation are focused, the diagram *plot* can be used as a *model for* statistical reasoning. Therefore, if complex statistical questions are to be answered with the diagram *plot*, it would be advisable for the diagram to become a *model for* statistical reasoning as quickly as possible. Here too, however, the learners must be trained in the usage of the diagram, otherwise, they will not be able to recognise the relationships in the diagram and will not be able to use it as a *model for* statistical reasoning. Suggestions on how to practise first the use of statistical diagrams with analogue materials and then to focus on interpreting larger amounts of data with digital materials can be found, for example, in Frischemeier (2018).

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